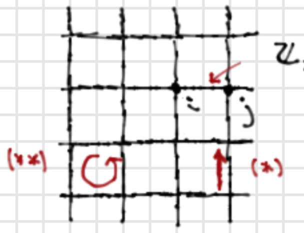
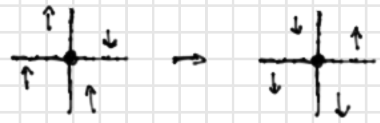
$\mathbb{Z}_2$-lattice gauge theory (Wegner 71, review Kogut RMP 79)

- $\mathbb{Z}_2$-degrees of freedom frequently emerging in correlated fermion systems (Senthil & Fisher, PRB 62, 7850 (2000)): $c^\dagger \to e^{i\phi}\cdot f^\dagger = e^{i\phi}(-1)\times(-1)\, f^\dagger$
  $c^\dagger$: fermion; $e^{i\phi}$: charge; $f^\dagger$: neutral fermion
  ⇝ an emergent 'gauge degree of freedom'
- Formulate $\mathbb{Z}_2$ gauge theory on a lattice. (Here 2d □-lattice for simplicity)



$\mathbb{Z}_2$- d.o.f. $|\pm\rangle_{ij}$ eigenstates of op $\sigma_{z,ij}$.

- local gauge trafo at i acts through $\prod_{n.n.\, i} \sigma_{x,ij}$
  ⇝ $|\pm\rangle_{ij} \to |\mp\rangle_{ij}$



- dynamical players of $\mathbb{Z}_2$ lattice gauge theory
  - gauge field along link $i\to j$: $\sigma_{z,ij}$ $\left(\hat{=}\ e^{iA_{ij}}\right.$ in $U(1)$ lattice ED$\left.\right)$
  - electric flux through link $i\to j$: $\sigma_{x,ij}$. (*) $\qquad \sigma_z\sigma_x\sigma_z = -\sigma_x \qquad \left(\hat{=}\ e^{iA}\, E\, e^{-iA} = E+1\right)$
  - magnetic flux through plaquette s: $\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}$. (**) $\quad \left(\hat{=}\ e^{iA_{ij}} \cdots e^{iA_{li}}\right)$

  electric and magnetic flux are gauge invariant

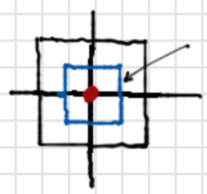
- gauge invariant Hamiltonian

$$\hat{H} = -\gamma \sum_{\text{links}} \overset{(*)}{\sigma_{x,ij}} - \lambda \sum_{\square} \overset{(**)}{\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}}$$

- system supports quantum phase transition driven by $r \equiv \lambda/\gamma$
  1) confining phase $r \lesssim 1$
  2) topological phase $r \gtrsim 1$

1) Confining phase

- Define 'charge density op': $\hat{\rho}_i = \prod_j \sigma_{x,ij}$
  Charge is locally conserved by $\hat{H}$: $[\hat{H}, \hat{\rho}_i] = 0$

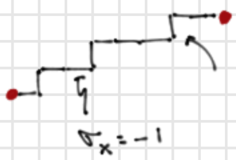


flip all spins cut by this dual lattice loop

- Ground state in charge neutral sector : $\forall i \quad \hat{\rho}_i |\Psi\rangle = 0$:
  $\sigma_{x,ij} = 1$ globally

• Q: What is ground state in sector of Hilbert space with two charges sitting at i,j?

A: $(\lambda = 0)$

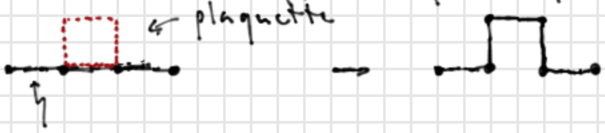


a minimal electric flux line. Costs energy $2\gamma\, \underbrace{d(i,j)}_{\text{Manhattan}} = \underbrace{\sigma}_{\text{string tension}}\, d(i,j)$

Energy grows linearly with distance: confinement

• Q: What is the effect of the flux term on this

A:



$\sim$ string fluctuations $\sigma \to 2\gamma - \underbrace{\lambda^2}_{\text{strength of pert.}} / \underbrace{4\gamma}_{\text{excitation energy}}$

magnetic flux term softens string tension

2) Spin liquid phase

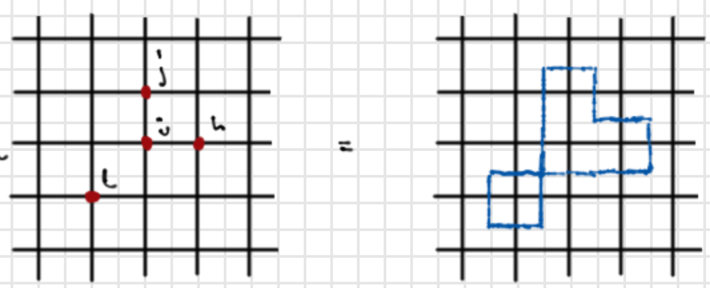
• Consider limit $\gamma = 0$. Ground state has flux $\prod_{\square} \tau_z \dots \tau_z$ on each plaquette. This is an implicit characterization.

• Q: How does the ground state in a sector of fixed charge look like?

A: Start from all spins $\tau_{z,ij} = 1$ state $|\Psi_p\rangle$. This is not a charge eigenstate

$\hat{p}_i$

$\hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l$

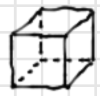


ground state: state of all equal weight superposition of closed strings (cf. BCS), a string net condensate

Corrolary: charges totally deconfined.

Q: Is the ground state unique?

A: Def.: V: no. of vertices, E: no. of edges, F: no. of faces

Compactify surface (for simplicity), e.g. . Counting:

E d.o.f. (the spins)

$-(V-1)$ charge constraints (-1 is overall charge neutrality)

$-(F-1)$ flux constraint (-1 is overall flux neutrality)

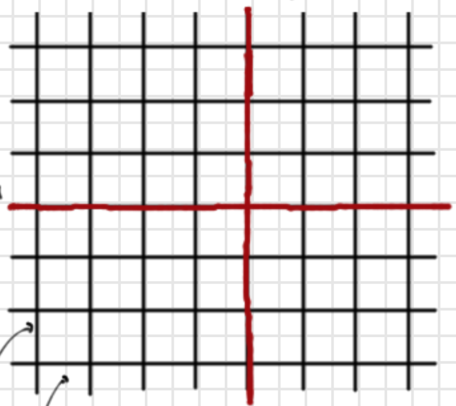
$\underbrace{-F+E-V}_{(-)\text{Euler-Characteristic}}-2$ qubit d.o.f. remain

(−) Euler-Characteristic $\chi = 2-2g$. Surface genus g.

$\leadsto$ ground state degeneracy: 2^{2g}. A hallmark of topological matter.

Q: How do we characterize different ground states?

A:

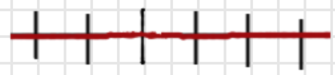
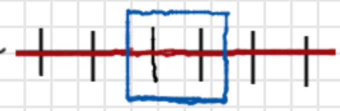


assume periodic boundary conditions a torus, T^2

$\sigma_z^a = \prod_{C_a} \sigma_{z,ij}$.

C_a or any other curve winding around T^2.

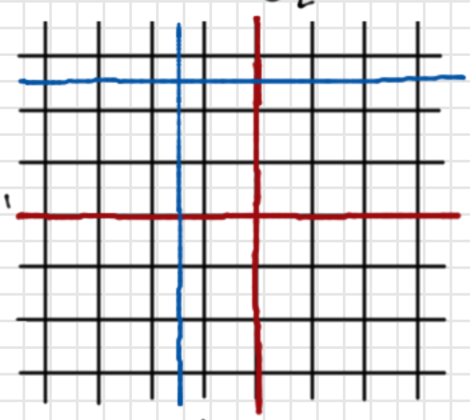
Claim: $\{\sigma_z^1, \sigma_z^2\}$ are topological invariants of each charge sector. E.g.

 and 

same C'

Invariants changed by nonlocal operators.

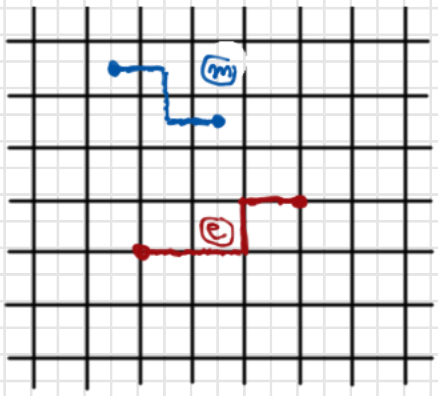
$\sigma_x^a = \prod_{C_a'} \sigma_{x,ij}$ $\qquad [\sigma_x^a, \sigma_z^a]_+ = 0$



Global qubits $\{\sigma_x^a, \sigma_z^a\}$ provide topological characterization of ground state $\leadsto$ topological quantum computation.

Q: What are excitations of the system?

A:



m: a 'magnetic' excitation $= \prod_{\substack{\text{all links}\\ \text{cut by string}}} \sigma_{x,ij}$

changes flux of terminal plaquettes. Costs energy 2λ.

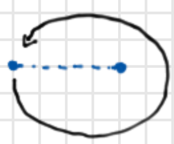
e: an 'electric excitation' $= \prod_{\substack{\text{all links along}\\ \text{path}}} \sigma_{z,ij}$

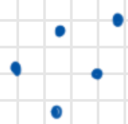
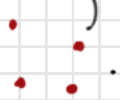
changes charge at terminal points. If system contains 'chemical potential' $\mu \cdot \sum_i \hat{p}_i$, energy cost 2μ.

Think of e, m as quantum particles forming on top of (closed) string ground state.

Q: What is the statistics of these particles?

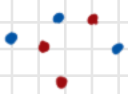
A: Find out what happens as we braid them around one other.

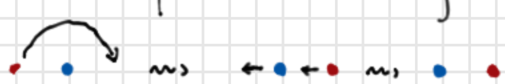


$\rightsquigarrow$ nothing happens  : a system of bosons. The same with .

However: 

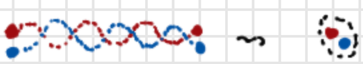
string of σ_x 'cuts' through one of σ_z: a minus sign

 are semions relative to each other. Note: particle exchange: a π-rotation



	π-exch.	2π-braid
fermions	-1	1
semions	i	-1

Something interesting happens if we fuse m and e into a composite excitation:

 are fermions relative to each other. Have generated fermions

as
- emergent particles of gauge theory
- terminal excitations of strings (cf. Jordan-Wigner in 1d)
- particles obeying strict parity conservation

$\rightsquigarrow$ conceptual proximity to string theory

- Summary:
  - $\mathbb{Z}_2$ gauge theory has phase transition without local order parameter or symmetry breaking
  - $\mathbb{Z}_2$ topological fluid = a string condensate
  - Rigorous ($L\to\infty$) ground state degeneracy 2^3 (the closest approx. to an 'order parameter')
  - Supports (abelian) quasi particles as excitations, including emergent fermions

- Appendix: U(1) lattice gauge theory (ideas)

Put electrodynamics on a 3+1 dim. lattice. Ingredients (Euclidean metric)

- gauge potential A_{ij} sits on links and enters through phases $e^{iA_{ij}}$
- gauge transformation ϕ_i acts on nodes via phases $e^{i\phi_i}$
- gauge invariant action: $S[A]\sim \sum_{\square} e^{iA_{ij}}e^{iA_{jk}}e^{iA_{kl}}e^{iA_{li}}$
  becomes $S_E[A]=\frac{1}{4}\int d^4x\; F_{\mu\nu}F^{\nu\mu}$ $\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ in continuum limit
  (and passage to Minkowski)
- Hamiltonian of lattice gauge theory

  Chose gauge $A_0=\varphi=0$. $S_c[A]=\frac{1}{2}\int d^4x\;\big(\partial_t A_i\,\partial_t A_i-(\nabla\times A)_i(\nabla\times A_i)\big)=$

  $$\left(=\frac{1}{2}\int d^4x\;\big(E_iE_i-B_iB_i\big)\right)$$

  Canonical momentum (of 'coordinate' A) $\delta S_c[A]/\delta\partial_t A_i=E_i$

  $\rightsquigarrow$ Hamiltonian $H[A,E]=\frac{1}{2}\int d^4x\,(E_i^2+B_i^2)\rightsquigarrow$ on a 3d (!) lattice:

  $$H=\sum_{\langle i,j\rangle}\hat{E}_{ij}^2+\sum_{\square}e^{i\hat{A}_{ij}}\cdots e^{i\hat{A}_{li}}\qquad [\hat{A}_{ij},\hat{E}_{ij}]=1$$

  this is the U(1) analog of the $\mathbb{Z}_2$-Hamiltonian discussed above.